Integrable particle systems and Macdonald processes

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Lecture 3

- Macdonald measure and process generalizes Schur process
- Structure of Macdonald polynomials leads to integrable particle systems (e.g. q-TASEP, stochastic heat and KPZ equations…)
- Eigenrelations satisfied by Macdonald polynomials leads to explicit formulas for expectations of observables and certain asymptotics

Review: Schur processes

$$S_{\lambda/\mu}(x_N) := x_N^{|\lambda|-|\mu|} \mathbb{1}_{\lambda \succeq \mu}$$

$$s_\lambda(x_1,\dots,x_N) := \frac{\det\left[x_i^{N+\lambda_j-j}\right]_{i,j=1}^N}{\det\left[x_i^{N-j}\right]_{i,j=1}^N} = \sum_{\mu \preceq \lambda} s_\mu(x_1,\dots,x_{N-1})\, x_N^{|\lambda|-|\mu|}$$

$$\mathbb{SM}_{X;Y}(\lambda) := \frac{s_\lambda(X)\, s_\lambda(Y)}{\Pi(X;Y)}$$

$$= \sum_\lambda s_\lambda(X) s_\lambda(Y) = \prod_{i,j} \frac{1}{1-x_i y_j}$$

$$\Lambda^k_{k-1}(\lambda,\mu) := \frac{s_\mu(x_1,\dots,x_{k-1})}{s_\lambda(x_1,\dots,x_{k-1},x_k)}\, s_{\lambda/\mu}(x_k)$$

$$\mathbb{S}_{X;Y}(\lambda^{(N)},\dots,\lambda^{(1)}) := \mathbb{SM}_{X;Y}(\lambda^{(N)})\, \Lambda^N_{N-1}(\lambda^{(N)},\lambda^{(N-1)}) \cdots \Lambda^1_0(\lambda^{(1)},\varnothing)$$

+ fixed level dynamics intertwining with Λ^k_{k-1}, and eigenrelations

We will upgrade all of this to full Macdonald setting

Macdonald processes $q, t \in [0,1)$

Ruijsenaars–Macdonald system

Representations of Double Affine Hecke Algebras

q-Whittaker processes $t=0$

q-TASEP, 2d dynamics

q-deformed quantum Toda lattice

Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$

General β RMT $t=q^{\beta/2}\to 1$

Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$

Calogero–Sutherland, Jack polynomials

Spherical functions for Riem. Symm. Sp.

Hall–Littlewood processes $q=0$

Random matrices over finite fields

Spherical functions for p-adic groups

Whittaker processes $t=0$, $q\to 1$

Directed polymers and their hierarchies

Quantum Toda lattice, repr. of $GL(n,\mathbb{R})$

Kingman partition structures $q=0$, $t=1$

Cycles of random permutations

Poisson–Dirichlet distributions

Schur processes $q=t$

Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE

Characters of symmetric, unitary groups

Defining Macdonald polynomials [Macdonald, 1987]

Macdonald polynomials $P_\lambda(x_1,\ldots,x_N) \in \mathbb{Q}(q,t)[x_1,\ldots,x_N]^{S(N)}$ with partitions $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_N \geq 0)$ form a basis in symmetric polynomials in N variables over $\mathbb{Q}(q,t)$. They diagonalize

$$(\mathcal{D}_1 f)(x_1,\ldots,x_N) = \sum_{i=1}^{n} \prod_{j \neq i} \frac{t x_i - x_j}{x_i - x_j} f(x_1,\ldots,q x_i,\ldots,x_N)$$

with (generically) pairwise different eigenvalues

$$\mathcal{D}_1 P_\lambda = (q^{\lambda_1} t^{N-1} + q^{\lambda_2} t^{N-2} + \ldots + q^{\lambda_N}) P_\lambda .$$

Remarkable properties: Orthogonality (dual basis Q_λ), simple Cauchy type identity, Pieri and branching rules, explicit generators of the algebra of operators commuting with $\mathcal{D}_1$, etc.

(Ascending) Macdonald processes

Probability measures on Gelfand–Tsetlin scheme (interlacing array)

$$\begin{array}{ccccccccc}
\lambda^{(N)}_N & & \lambda^{(N)}_{N-1} & & \cdots & & \lambda^{(N)}_2 & & \lambda^{(N)}_1 \\
& \lambda^{(N-1)}_{N-1} & & \lambda^{(N-1)}_{N-2} & \cdots & & & \lambda^{(N-1)}_1 & \\
& & \ddots & & \cdots & & \iddots & & \\
& & & \lambda^{(2)}_2 & & \lambda^{(2)}_1 & & & \\
& & & & \lambda^{(1)}_1 & & & &
\end{array}$$

$$\lambda^{(m)}_j \in \mathbb{Z}_{\geq 0}$$

Macdonald Measure

Macdonald polynomials

$$\mathbb{MM}_{\{a_i\};\{b_j\}}\left(\lambda^{(N)}\right) = \frac{P_{\lambda^{(N)}}(a_1, \dots, a_N)\, Q_{\lambda^{(N)}}(b_1, \dots, b_M)}{\Pi(\underbrace{a_1, \dots, a_N}; \underbrace{b_1, \dots, b_M})}$$

normalization constant

two groups of parameters

Cauchy type identity

$$\Pi(a_1,\dots,a_N;b_1,\dots,b_M) := \sum_{\lambda^{(N)}} P_{\lambda^{(N)}}(a_1,\dots,a_N)\, Q_{\lambda^{(N)}}(b_1,\dots,b_M)$$

$$= \prod_{i,j} \frac{(t a_i b_j; q)_\infty}{(a_i b_j; q)_\infty}$$

q-Pochhammer symbol

$$(a;q)_n := \prod_{i\geq 0}^{n-1} (1-q^i a)$$

Ex: If $b_i \equiv \frac{\gamma}{M}\frac{1-q}{1-t}$ and $M \to \infty$ then $\Pi(a;b) \to e^{\gamma a_1} \cdots e^{\gamma a_N}$ (Plancherel specialization)

At $q=t$ reduces to Schur function Cauchy identity

$$\sum_\lambda S_\lambda(a) S_\lambda(b) = \prod_{i,j} \frac{1}{1-a_i b_j}$$

Recovering Schur case

At $q=t$, $P_\lambda = Q_\lambda = s_\lambda$ and we recover Schur measure [Okounkov '01] which is a discrete version of a random matrix eigenvalue ensemble; and Schur Process [Okounkov-Reshetikhin '03] which is a discrete version of the GUE corner process.

Can think of Macdonald measure as a (q,t)-deformed discrete random matrix eigenvalue type ensemble (and analogously Macdonald process as deformed eigenvalue corner ensemble).

BUT: For general $q \neq t$ this is NOT DETERMINANTAL

Branching rule:

$$P_\lambda(a_1,\dots,a_{N-1},a_N) = \sum_{\mu \preceq \lambda} P_\mu(a_1,\dots,a_{N-1})\, P_{\lambda/\mu}(a_N)$$

Skew Macdonald polynomial

$$P_{\lambda/\mu}(u) = \begin{cases} \psi_{\lambda/\mu}\, u^{|\lambda|-|\mu|}, & \lambda \succeq \mu \\ 0, & \text{else} \end{cases} \qquad (\psi_{\lambda/\mu} \in \mathbb{Q}(q,t))$$

explicit

Combinatorial expansion shows for positive a's, $P_\lambda(a_1,\dots,a_k) \geq 0$

For example, when $t=0$:

$$\psi_{\lambda/\mu} = \frac{\prod_{i=1}^{N-1} (\lambda_i - \lambda_{i+1})!_q}{\prod_{i=1}^{N-1} (\lambda_i - \mu_i)!_q\, (\mu_i - \lambda_{i+1})!_q}$$

$$k!_q = \frac{(1-q)\cdots(1-q^k)}{(1-q)^k}$$

Ex: Prove $k!_q \xrightarrow{q \to 1} k!$

Gibbs property

Stochastic links from level N to $N-1$

$$\Lambda^{N}_{N-1}\left(\lambda^{(N)},\lambda^{(N-1)}\right) := \frac{P_{\lambda^{(N-1)}}(a_1,\dots,a_{N-1})\,P_{\lambda^{(N)}/\lambda^{(N-1)}}(a_N)}{P_{\lambda^{(N)}}(a_1,\dots,a_N)}$$

Maps $\mathbb{MM}_{\{a_1,\dots,a_N\};\{b_1,\dots,b_M\}}$ to $\mathbb{MM}_{\{a_1,\dots,a_{N-1}\};\{b_1,\dots,b_M\}}$.

Trajectory of this Markov chain defines the Macdonald process

$$\mathbb{M}_{\{a\},\{b\}}\left(\lambda^{(N)},\dots,\lambda^{(1)}\right) := \mathbb{MM}_{\{a\},\{b\}}\left(\lambda^{(N)}\right)\Lambda^{N}_{N-1}\left(\lambda^{(N)},\lambda^{(N-1)}\right)\cdots\Lambda^{1}_{0}\left(\lambda^{(1)},\emptyset\right)$$

We are able to do two basic things [Borodin-C, 2011]:

- Construct explicit Markov operators that map Macdonald processes to Macdonald processes (with new parameters)
- Evaluate averages of a rich class of observables

The integrable structure of Macdonald polynomials directly translates into probabilistic content.

By working at a high combinatorial level we avoid analytic issues (eventually need to work hard to take various limits).

Discrete time/space (q,t)-deformed Dyson Brownian motion

Markov chain on level N preserves class of Macdonald measure:

$$\pi_u^{(N)}\left(\lambda^{(N)}, \nu^{(N)}\right) := \frac{P_{\nu^{(N)}}(a_1,\dots,a_N)}{P_{\lambda^{(N)}}(a_1,\dots,a_N)} \cdot \frac{Q_{\nu^{(N)}/\lambda^{(N)}}(u)}{\Pi(a_1,\dots,a_N;u)}$$

$u \geq 0$

maps $\mathbb{MM}_{\{a_1,\dots,a_N\};\{b_1,\dots,b_M\}}$ to $\mathbb{MM}_{\{a_1,\dots,a_N\};\{b_1,\dots,b_M,u\}}$.

Markov Intertwining relation lets us construct (2+1)d dynamics

$$\begin{tikzcd}
\mathbb{MM}_{\{a_1,\dots,a_N\};\{b_1,\dots,b_M\}} \sim & \lambda^{(N)} \arrow[r, "\pi_u^{(N)}"] \arrow[d, "\Lambda^N_{N-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mu^{(N)} \arrow[d, "\Lambda^N_{N-1}"] & \sim \mathbb{MM}_{\{a_1,\dots,a_N\};\{b_1,\dots,b_M,u\}} \\
\mathbb{MM}_{\{a_1,\dots,a_{N-1}\};\{b_1,\dots,b_M\}} \sim & \lambda^{(N-1)} \arrow[r, "\pi_u^{(N-1)}"] & \mu^{(N-1)} & \sim \mathbb{MM}_{\{a_1,\dots,a_{N-1}\};\{b_1,\dots,b_M,u\}}
\end{tikzcd}$$

$$\begin{tikzcd}
\lambda^{(N)} \arrow[r, "\pi_u^{(N)}"] \arrow[d, "\Lambda^N_{N-1}"'] \arrow[dr, phantom, "\circlearrowleft"] & \nu^{(N)} \arrow[d, "\Lambda^N_{N-1}"] \\
\lambda^{(N-1)} \arrow[r, "\pi_u^{(N-1)}"] \arrow[d] & \nu^{(N-1)} \arrow[d] \\
\vdots \arrow[d, dashed] & \vdots \arrow[d, dashed] \\
\lambda^{(2)} \arrow[r, "\pi_u^{(2)}"] \arrow[d, "\Lambda^2_1"'] \arrow[dr, phantom, "\circlearrowleft"] & \nu^{(2)} \arrow[d, "\Lambda^2_1"] \\
\lambda^{(1)} \arrow[r, "\pi_u^{(1)}"] & \nu^{(1)}
\end{tikzcd}$$

Multivariate Markov kernel

$$\mathbb{P}_u(\lambda,\nu) := \pi_u^{(1)}(\lambda^{(1)},\nu^{(1)}) \prod_{k=2}^{N} \frac{\pi_u^{(k)}(\lambda^{(k)},\nu^{(k)})\,\Lambda^k_{k-1}(\nu^{(k)},\nu^{(k-1)})}{(\pi_u^{(k)}\Lambda^k_{k-1})(\lambda^{(k)},\nu^{(k-1)})}$$

sequentially updates GT-pattern, mapping

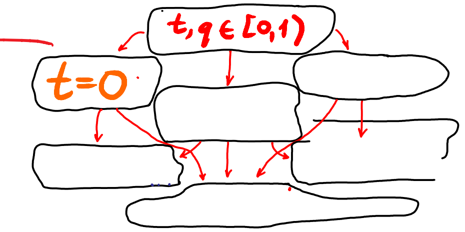
$$\mathbb{M}_{\{a_1,\dots,a_N\};\{b_1,\dots,b_M\}} \text{ to } \mathbb{M}_{\{a_1,\dots,a_N\};\{b_1,\dots,b_M,u\}}$$

Other dynamics may preserve class too

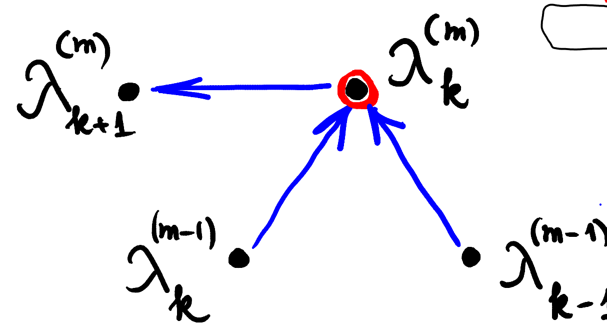
[O'Connell–Pei, 2012; Borodin–Petrov, 2013]

A new particle system: q-TASEP

Here is an example of a Markov process preserving the class of the q-Whittaker processes (Macdonald processes with $t=0$).



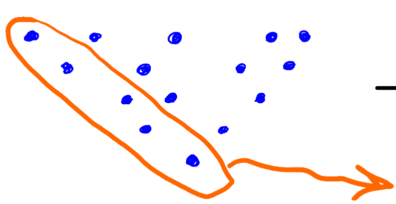
Each coordinate jumps by 1 to the right independently of the others with

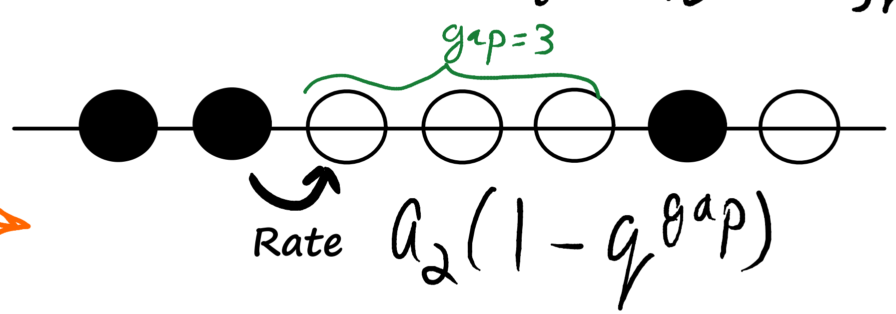


$$\mathrm{rate}\left(\lambda^{(m)}_k\right) = a_m \frac{\left(1-q^{\lambda^{(m-1)}_{k-1}-\lambda^{(m)}_k}\right)\left(1-q^{\lambda^{(m)}_k-\lambda^{(m)}_{k+1}+1}\right)}{\left(1-q^{\lambda^{(m)}_k-\lambda^{(m-1)}_k}\right)}$$

simulation

The set of coordinates $\left\{\lambda^{(m)}_m - m\right\}_{m\geq 1}$ forms q-TASEP





This is how we first discovered q-TASEP!

Particle systems described by (limits of) t=0 Macdonald processes

Discrete time q-TASEPs

q-pushASEP

q-TASEP

log-Gamma discrete polymer

semi-discrete Brownian polymer

KPZ/SHE/continuous Brownian polymer

universal limits (Tracy-Widom distributions, Airy processes)

+ previously studied systems arising from Schur processes (TASEP, LPP, tilings, plane partitions, GUE)

Evaluation of averages of rich class of observables

Let $\mathcal{D}$ be an operator that is diagonalized by the Macdonald polynomials (for example, a product of Macdonald operators),

$$\mathcal{D} P_\lambda = d_\lambda P_\lambda .$$

Applying it to the Cauchy type identity $\sum_\lambda P_\lambda(a) Q_\lambda(b) = \Pi(a;b)$ we obtain

$$\mathbb{E}[d_\lambda] = \frac{\mathcal{D}^{(a)} \Pi(a;b)}{\Pi(a;b)} .$$

If all the ingredients are explicit (as for products of Macdonald operators), we obtain meaningful probabilistic information. Contrast with the lack of explicit formulas for the Macdonald polynomials.

Macdonald difference operators

$$\mathcal{D}_N^r := t^{r(r-1)/2} \sum_{\substack{I\subset\{1,\dots,N\}\\ |I|=r}} \prod_{\substack{i\in I\\ j\notin I}} \frac{tx_i - x_j}{x_i - x_j} \prod_{i\in I} T_{q,x_i}$$

$r = 1, \dots, N$

$(T_{q,x_i} f)(x_1,\dots,x_N) = f(x_1,\dots, q x_i, \dots, x_N)$

Ex: Relate at $t=q$ to Schur q-difference operators

Commuting operators all diagonalized by $\{P_\lambda\}_{\ell(\lambda)\le N}$

$$\mathcal{D}_N^r P_\lambda(x_1,\dots,x_N) = e_r(q^{\lambda_1}t^{N-1}, q^{\lambda_2}t^{N-2}, \dots, q^{\lambda_N}t^0) P_\lambda(x_1,\dots,x_N)$$

$e_r(y_1,\dots,y_N) = \sum_{i_1<\dots<i_r} y_{i_1}\cdots y_{i_r}$

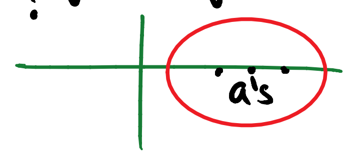
Expectations characterize Macdonald measure

$$\mathbb{E}\left[\prod_{i=1}^{k} e_{r_1}(q^{\lambda_i}t^{N-i})\right] = \frac{\mathcal{D}_N^{r_1}\cdots\mathcal{D}_N^{r_k}\,\Pi(a_1,\dots a_N; b_1,\dots,b_M)}{\Pi(a_1,\dots a_N; b_1,\dots,b_M)}$$

Encoding difference operators as contour integrals

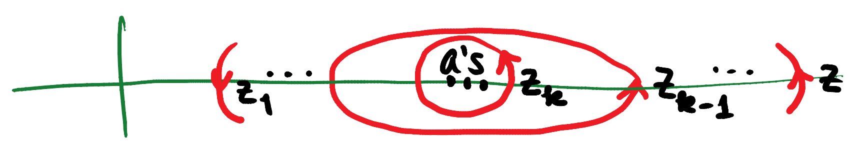
For (nice) multiplicative functions $F(u_1,\ldots,u_N) = f(u_1)\cdots f(u_N)$

$$(\mathcal{D}_N^r F)(\vec{a}) = \frac{F(\vec{a})}{(2\pi i)^r r!}\int\cdots\int \det\left(\frac{1}{tz_k - z_l}\right)_{k,l=1}^{r} \prod_{j=1}^{r}\left(\prod_{m=1}^{N}\frac{tz_j - a_m}{z_j - a_m}\right)\frac{f(qz_j)}{f(z_j)}\,dz_j$$



Another example for powers of first difference operators at $t=0$

$$((\mathcal{D}_N^1)^k F)(\vec{a}) = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\int\cdots\int \prod_{1\le A<B\le k}\frac{z_A - z_B}{z_A - qz_B}\prod_{j=1}^{k}\left(\prod_{m=1}^{N}\frac{a_m}{a_m - z_j}\right)\frac{f(qz_j)}{f(z_j)}\frac{dz_j}{z_j}$$



Applying to Macdonald process at t=0

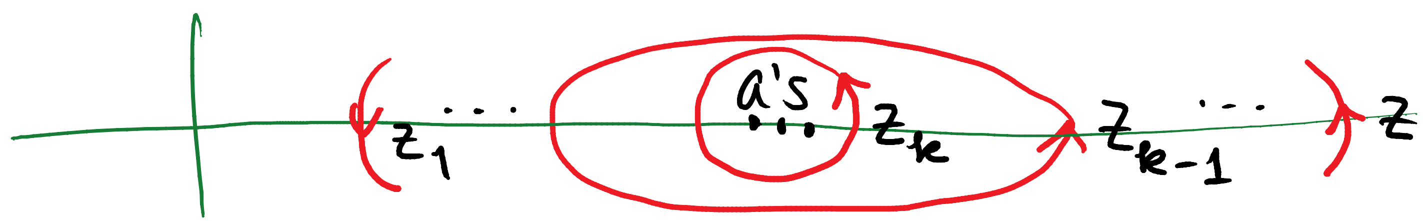
Taking t=0, we have that $\mathcal{D}_N^1 P_\lambda = q^{\lambda_N} P_\lambda$

Note $\Pi(a_1,\ldots,a_N; b_1,\ldots,b_M) = \Pi(a_1; b_1,\ldots,b_M)\cdots\Pi(a_N; b_1,\ldots,b_M)$

So taking products of the first order Macdonald operators (on different levels) results in the integral representation

$$\mathbb{E}\left[\prod_{i=1}^{k} q^{\lambda_{N_i}^{(N_i)}}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k}\int\cdots\int \prod_{1\le A<B\le k}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k}\left(\prod_{m=1}^{N}\frac{a_m}{a_m-z_j}\right)\frac{\Pi(qz_j;b)}{\Pi(z_j;b)}\frac{dz_j}{z_j}$$

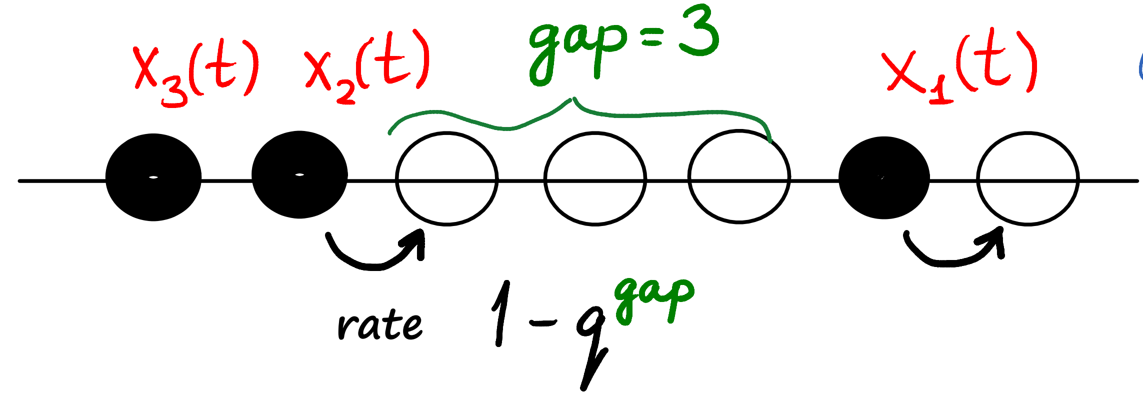
$(N_1 \geq N_2 \geq \cdots \geq N_k)$



Application to q-TASEP

q-TASEP corresponds to $t=0$ and $\Pi(a_1,\dots,a_N : \underbrace{\varepsilon,\dots,\varepsilon}_{\varepsilon^{-1}t \text{ times}}) = \prod_{i=1}^{N} e^{a_i t}$

Taking all $a_i \equiv 1$ and
recalling that $q^{\lambda_m^{(m)}} = q^{x_m(t)+m}$



$0<q<1$

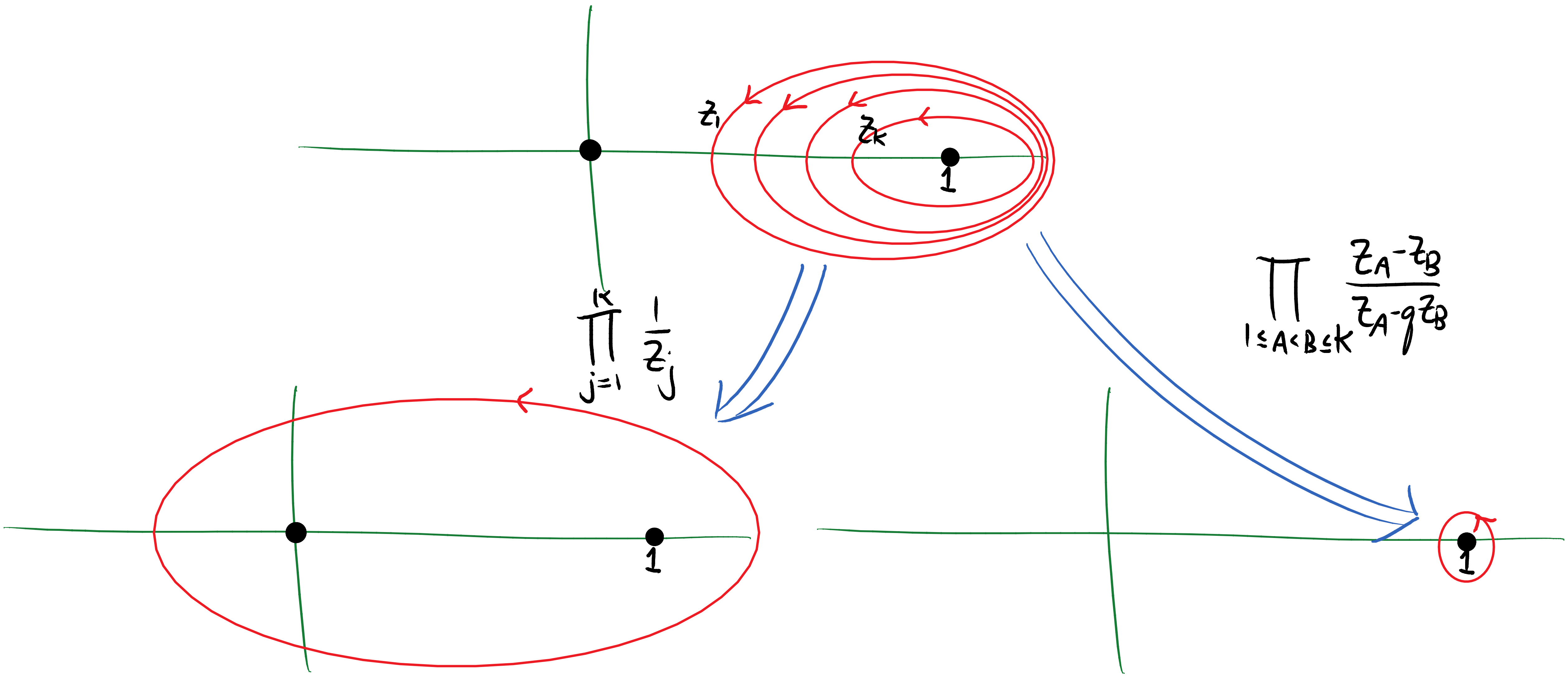
<u>Theorem</u>: *For q-TASEP with step initial data* $\{x_n(0)=-n\}_{n\geq 1}$

$$\mathbb{E}\left[\prod_{i=1}^{k} q^{(x_{N_i}(\tau)+N_i)}\right] = \frac{(-1)^k q^{\frac{k(k-1)}{2}}}{(2\pi i)^k} \oint \cdots \oint \prod_{A<B} \frac{z_A - z_B}{z_A - q z_B} \prod_{j=1}^{k} \frac{e^{(q-1)\tau z_j}}{(1-z_j)^{N_j}} \frac{dz_j}{z_j}$$

$(N_1 \geq N_2 \geq \dots \geq N_k)$

$*\ 0 \quad z_1 \cdots \quad 1 \ z_k \quad z_{k-1} \cdots \quad z_1$ (z_A contains $q z_B$, $B>A$)

Deforming nested contours together

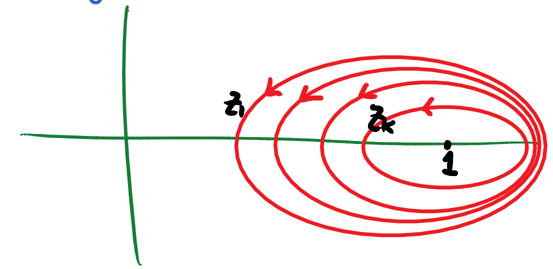


Must account for residues from poles crossed in deformation

Small contour expansion

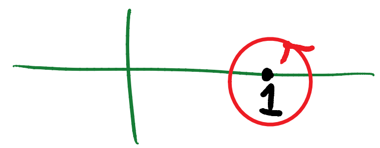
eg $f(z)=e^{tz}$ for q-TASEP

$$\frac{(-1)^k q^{k(k-1)/2}}{(2\pi i)^k}\oint\cdots\oint \prod_{1\le A<B\le k}\frac{z_A-z_B}{z_A-qz_B}\prod_{j=1}^{k} \frac{1}{(1-z_j)^{n_j}}\,\frac{f(qz_j)}{f(z_j)}\,\frac{dz_j}{z_j}$$



$$=\sum_{\lambda\vdash k}\frac{(1-q)^k}{m_1!\,m_2!\cdots}\cdot\frac{1}{(2\pi i)^{\ell(\lambda)}}\oint\cdots\oint\det\left[\frac{1}{w_i q^{\lambda_i}-w_j}\right]_{i,j=1}^{\ell(\lambda)} E_{\vec n}(\vec w\circ\lambda)\,dw_1\cdots dw_{\ell(\lambda)}$$

$\lambda=1^{m_1}2^{m_2}\cdots$



where $\vec w\circ\lambda=\left(w_1, qw_1,\ldots, q^{\lambda_1-1}w_1,\ldots, w_{\ell(\lambda)}, qw_{\ell(\lambda)},\ldots, q^{\lambda_{\ell(\lambda)}-1}w_{\ell(\lambda)}\right)$,

$$E_{\vec n}(z_1,\ldots,z_k)=\prod_{j=1}^{k}\frac{f(qz_j)}{f(z_j)}\cdot\sum_{\sigma\in S_k}\prod_{k\ge B>A\ge 1}\frac{z_{\sigma(A)}-qz_{\sigma(B)}}{z_{\sigma(A)}-z_{\sigma(B)}}\prod_{j=1}^{k}\frac{1}{(1-z_{\sigma(j)})^{n_j}}$$

Moments of q-TASEP

For all $n_j \equiv n$, $E_{\vec{n}}$ simplifies to

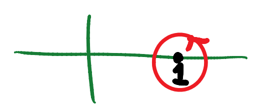
$$E_n(z_1,\dots,z_k) = \prod_{j=1}^{k} \frac{f(qz_j)}{f(z_j)} \cdot \frac{1}{(1-z_j)^n} \underbrace{\sum_{\sigma\in S_k} \prod_{k\geq B>A\geq 1} \frac{z_{\sigma(A)}-qz_{\sigma(B)}}{z_{\sigma(A)}-z_{\sigma(B)}}}$$

Ex: Prove that $C_k = \dfrac{(q;q)_k}{(1-q)^k} =: k_q!$

Conclusion: for step initial condition q-TASEP

$$f(z) = e^{tz}$$

$$\mathbb{E}^{\text{step}}\left[q^{k(x_n(t)+n)}\right] = k_q! \sum_{\substack{\lambda\vdash k \\ \lambda = 1^{m_1}2^{m_2}\dots}} \frac{(1-q)^k}{m_1!m_2!\cdots} \cdot \frac{1}{(2\pi i)^{\ell(\lambda)}} \oint\cdots\oint \det\left(\frac{1}{w_i q^{\lambda_i}-w_j}\right)_{i,j=1}^{\ell(\lambda)} \prod_{j=1}^{\ell(\lambda)} \frac{f(q^{\lambda_j}w_j)}{f(w_j)} \left(\frac{1}{(w_j;q)_{\lambda_j}}\right)^n$$



Moment generating function as a Fredholm determinant

$$\sum_{k\geq 0} \mathbb{E}^{\text{step}}\left[q^{k(x_n(t)+n)}\right] \frac{\zeta^k}{k_q!} = \det(I + K_\zeta)_{L^2(\circ)}$$

for $|\zeta|$ small enough. Here

$f(z) = e^{tz}$

$$K_\zeta(w,w') = \sum_{\lambda=1}^{\infty} \left[(1-q)\zeta\right]^\lambda \frac{f(q^\lambda w)\,(q^\lambda w;q)_\infty^N}{f(w)\,(w;q)_\infty^N} \cdot \frac{1}{wq^\lambda - w'}$$

"Mellin Barnes" representation suitable for asymptotics

$$K_\zeta(w,w') = \frac{1}{2\pi i}\int_{\frac{1}{2}+i\mathbb{R}} \frac{\pi}{\sin(-\pi s)} \left[-(1-q)\zeta\right]^s \frac{f(q^s w)\,(q^s w;q)_\infty^N}{f(w)\,(w;q)_\infty^N} \cdot \frac{1}{wq^s - w'}\, ds$$

<u>q-Laplace transform Fredholm determinant</u>

Moments of $q^{x_n(t)+n} \leq 1$, so they characterize the distribution.

q-deformed exponential [Hahn '49]:

$$e_q(x) := \frac{1}{((1-q)x;q)_\infty} = \sum_{k=0}^{\infty} \frac{x^k}{k_q!}$$

Ex: Prove $e_q(x) \to e^x$ as $q \nearrow 1$

<u>Theorem</u>: *For q-TASEP with step initial data* $\{x_n(0)=-n\}_{n\geq 1}$

$$\mathbb{E}^{\text{step}}\left[e_q\left(\zeta q^{x_n(t)+n}\right)\right] = \sum_{k=0}^{\infty} \mathbb{E}^{\text{step}}\left[q^{k(x_n(t)+n)}\right]\frac{\zeta^k}{k_q!} = \det\left(I + K_\zeta\right)_{L^2}$$

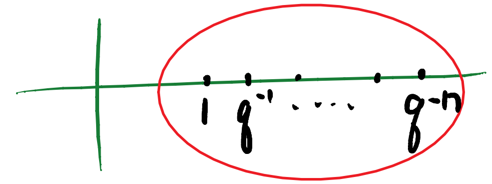
Some properties of q-Laplace transform

For $g \in \ell_1(\mathbb{N})$ and $z \in \mathbb{C} / q^{-\mathbb{N}}$ define $\hat{g}^q(z) := \sum_{n=0}^{\infty} \frac{g(n)}{(zq^n;q)_\infty}$

Ex: Prove inversion formula:

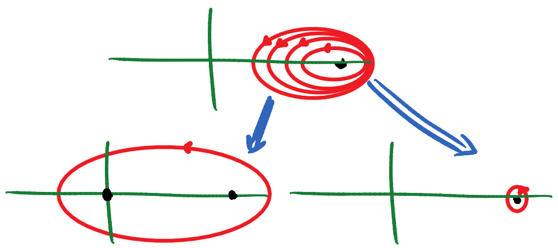
(hint: residues)

$$g(n) = -q^n \frac{1}{2\pi i} \oint (q^{n+1} z; q)_\infty \, \hat{g}^q(z) \, dz$$



Many other nice properties: linearity, scaling, shifts, transformation under q-derivative/integral, q-product/convolution relation, useful for solving q-difference equations [Bangerezako, 2008].

Large contour Fredholm determinant



Theorem: *For q-TASEP with step initial data* $\{x_n(0)=-n\}_{n\geq 1}$

$$\mathbb{E}^{\text{step}}\left[e_q\left(\zeta q^{x_n(t)+n}\right)\right] = \frac{1}{((1-q)\zeta;q)_\infty}\det(I+\zeta K)_{L^2(\;)}$$

$f(z) = e^{tz}$

$$K(w,w') = \frac{1-q}{(1-w)^N}\,\frac{f(qw)}{f(w)}\,\frac{1}{qw'-w}$$

"Cauchy" type formula simpler than "Mellin Barnes" type;

but apparently harder for asymptotic analysis.

Lecture 3 summary

- Macdonald measure and process generalizes Schur process
- Structure of Macdonald polynomials leads to integrable particle systems (e.g. q-TASEP, stochastic heat and KPZ equations...)
- Eigenrelations satisfied by Macdonald polynomials leads to explicit formulas for expectations of observables and certain asymptotics

Lecture 4 preview

- Expectations of q-TASEP observables solve integrable many body systems which can be solved via variant of Bethe ansatz
- Limit to directed polymers shows this is rigorous replica method
- Also applies to discrete q-TASEPs, q-PushASEP, and ASEP

Macdonald processes $q, t \in [0,1)$

Ruijsenaars-Macdonald system

Representations of Double Affine Hecke Algebras

q-Whittaker processes $t=0$

q-TASEP, 2d dynamics

q-deformed quantum Toda lattice

Representations of $\widehat{\mathfrak{gl}}_N$, $U_q(\mathfrak{gl}_N)$

General β RMT $t=q^{\beta/2} \to 1$

Random matrices over $\mathbb{R}, \mathbb{C}, \mathbb{H}$

Calogero-Sutherland, Jack polynomials

Spherical functions for Riem. Symm. Sp.

Hall-Littlewood processes $q=0$

Random matrices over finite fields

Spherical functions for p-adic groups

Whittaker processes $t=0$, $q \to 1$

Directed polymers and their hierarchies

Quantum Toda lattice, repr. of $GL(n,\mathbb{R})$

Kingman partition structures

Cycles of random permutations $q=0$, $t=1$

Poisson-Dirichlet distributions

Schur processes $q=t$

Plane partitions, tilings/shuffling, TASEP, PNG, last passage percolation, GUE

Characters of symmetric, unitary groups